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Mixing Coefficient, Generalized Maximal Correlation Coefficients, and Weakly Positive Measures

MARISELA DOMÍNGUEZ

Universidad Central de Venezuela, Caracas, Venezuela

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Let μ be a positive finite Borel measure on the real line R. For $t \geqslant 0$ let $e_t \cdot E_1$ and E_2 denote, respectively, the linear spans in $L^2(R, \mu)$ of $\{e^{isx}, s > t\}$ and $\{e^{isx}, s < 0\}$. Let $\theta: R \to C$ such that $|\theta| = 1$, denote by $\alpha_t(\theta, \mu)$ the angle between $\theta \cdot e_t \cdot E_1$ and E_2. The problems considered here are that of describing those measures μ for which (1) $\alpha_t(\theta, \mu) > 0$, (2) $\alpha_t(\theta, \mu) \to \pi/2$ as $t \to \infty$ (such μ arise as the spectral measures of strongly mixing stationary Gaussian processes), and (3) give necessary and sufficient conditions for the rate of convergence of the generalized maximal correlation coefficient: $\rho_t(\theta, \mu) = \cos \alpha_t(\theta, \mu)$. Using this coefficient we characterize the stationary continuous processes that are (a) completely regular and (b) strongly mixing Gaussian. We also give necessary and sufficient conditions for the rate of convergence of (a) the maximal correlation coefficient and (b) the mixing coefficient in the Gaussian case. © 1992 Academic Press, Inc.

1. INTRODUCTION

The problems considered here arose in a natural way when we studied the following prediction theory problems: (a) characterize the spectrum of a strongly mixing Gaussian process in the continuous case (this problem was proposed by Yaglom [17] and solved by Helson and Sarason [12] for the discrete case); (b) characterize the continuous parameter stationary completely linearly regular process such that the maximal correlation coefficient ρ_t is $O(e^{-\lambda t})$ proposed by Ibragimov (in a private communication). Problem (a) was solved by Hayashi [11], who proves a theorem which is stated without proof by Ibragimov [14]. In [5] we solved this problem for tempered measures of order $\leqslant b$. Problem (b) has been partially solved by Ibragimov [14, 15], who gave some sufficient conditions for the rate of convergence of ρ_t; in [5] we presented necessary and sufficient conditions for $\rho_t = O(r_T)$ for given $r_t \to 0$ $(t \to \infty)$ and we also presented a generaliza-

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tion for tempered measures of order $\leqslant b$. In Sections 5 and 6 we present these results as particular cases of our characterization.

In Section 3 we introduce weakly positive measures and the lifting theorem; the first version of this theorem is discrete and is due to Cotlar and Sadosky [4]. Arocena and Cotlar gave a continuous version [1]. These notions were used in [6] for proving weighted inequalities for the Hilbert transform, with tempered measures.

In Section 4 the results are announced for a general function θ; in Section 5 for $b \geqslant 0$ we take $\theta(x) = ((x-i)/(x+i))^{b/2}$; this leads to results for tempered measures of order $\leqslant b$. Finally, in Section 6 we take $\theta(x) = 1$, and this leads to results for finite measures (these results were given in [5]).

2. Basic Problems

We use the following notation:

$$M(R) = \{\text{positive finite Borel measures in } R\}$$

$$e_t\colon R \to R, \qquad e_t(x) = e^{itx}$$

$$E = \operatorname{span}\{e_t,\ t \in R\}$$

$$E_1 = \operatorname{span}\{e_s,\ s > 0\} \qquad E_2 = \operatorname{span}\{e_s,\ s < 0\}.$$

Let $\theta\colon R \to C$ such that $|\theta| = 1$. For $\mu \in M(R)$, we define the generalized maximal correlation coefficient $\rho_t(\theta, \mu)$ as the cosine of the angle $\alpha_t(\theta, \mu)$ between the subspaces of $L^2(R, \mu)$: $\theta \cdot e_t \cdot E_1$ and E_2, that is,

$$\rho_t(\theta, \mu) = \sup_{\phi_1 \in E_1;\, \phi_2 \in E_2;\, \int_{-\infty}^{\infty} |\phi_1|^2\, d\mu = \int_{-\infty}^{\infty} |\phi_2|^2\, d\mu = 1} \left| \int_{-\infty}^{\infty} e_t \theta \phi_1 \overline{\phi_2}\, d\mu \right|.$$

So we shall study the following problems:

(1) Characterization of the $\mu \in M(R)$ such that $\rho_t(\theta, \mu) \leqslant r$ for $r \in (0, 1)$.

(2) Characterization of the $\mu \in M(R)$ such that $\lim_{t \to \infty} \rho_t(\theta, \mu) = 0$.

(3) Characterization of the $\mu \in M(R)$ such that $\rho_t(\theta, \mu) = O(r_t)$ with given $r_t \to 0$ $(t \to \infty)$.

For solving this problems we shall make use of the lifting theorem for weakly positive measures of measures in R.

In [7] we used a matricial version of this coefficient to explore the properties of the so-called matricially weakly positive measures and relate them with the invertibility of systems of Toeplitz operators, giving a matricial extension of the Widom–Devinatz theorem.

3. Weakly Positive Measures and Lifting Theorem

We generally follow the notation of [1, 2]. Let $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ be a matrix of finite measures in R.

Definition. $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive if for every Borel set $\Delta \subset R$ $(\mu_{\alpha\beta}(\Delta))_{\alpha,\beta=1,2}$ is a positive definite numerical matrix.

Proposition 3.1 [1, 2, 4]. *The following conditions are equivalent*:

(1) $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *is positive.*

(2) $\mu_{11} \geqslant 0$, $\mu_{22} \geqslant 0$, $\mu_{12} = \overline{\mu_{21}}$, *and* $|\mu_{12}(\Delta)|^2 \leqslant \mu_{11}(\Delta)\, \mu_{22}(\Delta)$ *for every Borel set* $\Delta \subset R$.

(3) $\mu_{11} \geqslant 0$, $\mu_{22} \geqslant 0$, $\mu_{12} = \overline{\mu_{21}}$, *and for all* $(\phi_1, \phi_2) \in E \times E$,

$$\left| \int_{-\infty}^{\infty} \phi_1 \overline{\phi_2}\, d\mu_{12} \right|^2 \leqslant \left(\int_{-\infty}^{\infty} |\phi_1|^2\, d\mu_{11} \right) \left(\int_{-\infty}^{\infty} |\phi_2|^2\, d\mu_{22} \right).$$

A weaker condition is the following.

Definition [1, 2, 4]. $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is weakly positive if it satisfies

$$\mu_{11} \geqslant 0, \qquad \mu_{22} \geqslant 0, \qquad \mu_{12} = \overline{\mu_{21}},$$

and for all $(\phi_1, \phi_2) \in E_1 \times E_2$,

$$\left| \int_{-\infty}^{\infty} \phi_1 \overline{\phi_2}\, d\mu_{12} \right|^2 \leqslant \left(\int_{-\infty}^{\infty} |\phi_1|^2\, d\mu_{11} \right) \left(\int_{-\infty}^{\infty} |\phi_2|^2\, d\mu_{22} \right).$$

The next theorem is the lifting property for weakly positive matrices of measures in R.

Let $H^1(R)$ be the space of Hardy functions in the upper half plane [10].

Theorem 3.2 [1, 2, 4]. *Let* $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *be a matrix of finite Borel measures in* R, *then the following conditions are equivalent*:

(a) $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *is weakly positive.*

(b) *There exists* $h \in H^1(R)$ *such that*

$$(\mu_{\alpha\beta})_{\alpha,\beta=1,2} + \begin{bmatrix} 0 & h\, dx \\ \bar{h}\, dx & 0 \end{bmatrix}$$

is positive.

We obtain a characterization of a class of weakly positive matrix measures, which is the crucial tool for solving our problems.

4. Characterization of the Generalized Maximal Correlation Coefficient

PROPOSITION 4.1. *Let $\theta\colon R \to C$ such that $|\theta| = 1$, let $\mu \in M(R)$, and $0 < r < 1$.*

$$\begin{bmatrix} r\mu & e_t\theta\mu \\ e_{-t}\bar{\theta}\mu & r\mu \end{bmatrix}$$

is weakly positive if and only if $\rho_t(\theta, \mu) \leqslant r$.

In this case, μ is an absolutely continuous measure.

Proof. $\rho_t(\theta, \mu) \leqslant r$ if and only if for all $(\phi_1, \phi_2) \in E_1 \times E_2$,

$$\left|\int_{-\infty}^{\infty} \phi_1 \overline{\phi_2} e_t \theta \, d\mu\right|^2 \leqslant \left(\int_{-\infty}^{\infty} |\phi_1|^2 \, r \, d\mu\right)\left(\int_{-\infty}^{\infty} |\phi_2|^2 \, r \, d\mu\right).$$

From the lifting property there exists $h \in H^1(R)$ such that for every borel set $\Delta \subset R$

$$\left|e_t\theta\mu(\Delta) + \int_\Delta h(x)\, dx\right|^2 \leqslant (r\mu(\Delta))^2.$$

If $|\Delta| = 0$ then $|e_t\theta\mu(\Delta)| \leqslant r\mu(\Delta)$, therefore $\mu(\Delta) = 0$.

THEOREM 4.2. *Let $\mu \in M(R)$, $r \in (0, 1)$, $t \geqslant 0$, and $\theta\colon R \to C$, such that $|\theta| = 1$ a.e. in R; then the following properties are equivalent:*

(a) $\rho_t(\theta, \mu) \leqslant r$.

(b) *There exists $h \in H^1(R)$ such that $d\mu = |h(x)|\, e^{u(x)}\, dx$ and $v = -\arg(e_{-t}\bar{\theta}h)$ satisfy*

$$|u(x)| \leqslant \operatorname{arccosh}\left(\frac{\cos v(x)}{\sqrt{1-r^2}}\right), \qquad |v(x)| \leqslant \frac{\pi}{2} - \arccos r < \frac{\pi}{2}.$$

Proof. $\rho_t(\theta, \mu) \leqslant r$ if and only if

$$\begin{bmatrix} r\mu & e_t\theta\mu \\ e_{-t}\bar{\theta}\mu & r\mu \end{bmatrix}$$

is weakly positive if and only if

$$\begin{bmatrix} rw\, dx & e_t\theta w\, dx \\ e_{-t}\bar{\theta}w\, dx & rw\, dx \end{bmatrix}$$

is weakly positive, where $d\mu = w(x)\,dx$ if and only if there exists $h \in H^1(R)$ such that for almost all $x \in R$,

$$|e_t(x)\,\theta(x)\,w(x) - \sqrt{1-r^2}\,h(x)|^2 \leqslant r^2 w^2(x).$$

Let $\phi = e_{-t}\bar{\theta}h$ and $v = -\arg\phi$. We have

$$\begin{aligned}&|e_t\theta w - \sqrt{1-r^2}\,h|^2 - r^2w^2\\&\quad= (1-r^2)\,w^2 - 2\sqrt{1-r^2}(\operatorname{Re}\phi)\,w + (1-r^2)\,|\phi|^2.\end{aligned}$$

Then $w_1 \leqslant w \leqslant w_2$, where for $k = 1, 2$ we have set

$$w_k = |\phi|\exp\left((-1)^k \operatorname{arccosh}\left(\frac{\cos v}{\sqrt{1-r^2}}\right)\right)$$

because for any $x \in R$ $(-1)^k \operatorname{arccosh}(x) = \log(x + (-1)^k\sqrt{x^2-1})$. Finally, $w_1 \leqslant w \leqslant w_2$ if and only if

$$\left|\log\left(\frac{w}{|\phi|}\right)\right| \leqslant \operatorname{arccosh}\left(\frac{\cos v}{\sqrt{1-r^2}}\right).$$

Let $u(x) = \log(w\,|\phi|^{-1})$. Then we have the first inequality and $w(x) = |h(x)|\,e^{u(x)}$.

Now, $|e_t\theta w - h| \leqslant rw$ implies

$$\left|\frac{1}{r} - \frac{e_{-t}\bar{\theta}h}{rw}\right| = \left|\frac{1}{r}e_t\theta - \frac{h}{rw}\right| \leqslant 1.$$

Then

$$v = -\arg\left(\frac{\phi}{rw}\right), \qquad \left|\frac{1}{r} - \frac{\phi}{rw}\right| \leqslant 1.$$

Thus $|v| < \pi/2$, and $\sin|v| \leqslant r$.

THEOREM 4.3. *Let $\mu \in M(R)$ and $\theta\colon R \to C$, such that $|\theta| = 1$ a.e. in R. Then the following properties are equivalent*:

(a) $\lim_{t\to\infty} \rho_t(\theta, \mu) = 0$.

(b) *For every $\varepsilon > 0$, there exists $t > 0$ and $h_\varepsilon \in H^1(R)$ such that*

$$d\mu = |h_\varepsilon(x)|\,e^{u_\varepsilon(x)}\,dx, \qquad v_\varepsilon = -\arg(e_{-t}\bar{\theta}h_\varepsilon),$$

and $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$.

Proof. (a)$\to$(b) Given $\varepsilon>0$, there exist $r_\varepsilon = r \in (0, 1)$ such that $\operatorname{arccosh}(1/\sqrt{1-r^2}) < \varepsilon/2$ and $\pi/2 - \arccos r < \varepsilon/2$.

By hypothesis there exist $t>0$ such that $\rho_t(\theta, \mu) \leqslant r$.

From Theorem 4.2, μ has the required representation,

$$|u_\varepsilon(x)| \leqslant \operatorname{arccosh}\left(\frac{\cos v_\varepsilon(x)}{\sqrt{1-r^2}}\right), \qquad |v_\varepsilon(x)| \leqslant \frac{\pi}{2} - \arccos r.$$

Therefore $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$.

(b)$\to$(a) As

$$\lim_{x\to 0} \operatorname{arccosh}\left(\frac{\cos x}{\sqrt{1-r^2}}\right) = \operatorname{arccosh}\left(\frac{1}{\sqrt{1-r^2}}\right) = L > 0$$

for $r>0$, there exist $\delta>0$ such that for $|x| \leqslant \delta$

$$\operatorname{arccosh}\left(\frac{\cos x}{\sqrt{1-r^2}}\right) > \frac{L}{2} > 0.$$

Let $\varepsilon = \min\{\delta; L/2\}$, then $\varepsilon < L/2 < \operatorname{arccosh}(\cos \varepsilon/\sqrt{1-r^2})$. And $\varepsilon < \arcsin r$ (when we define arccosh we take $\cos\varepsilon > \sqrt{1-r^2}$).

By hypothesis, there exist $t>0$ such that μ has the representation with $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$. So

$$\|v_\varepsilon\|_\infty < \varepsilon < \pi/2 - \arccos r$$

$$|u_\varepsilon| < \varepsilon < \operatorname{arccosh}\left(\frac{\cos\varepsilon}{\sqrt{1-r^2}}\right) < \operatorname{arccosh}\left(\frac{\cos v_\varepsilon}{\sqrt{1-r^2}}\right).$$

From Theorem 4.2, $\rho_t(\theta, \mu) \leqslant r$. (As $\rho_t(\theta, \mu)$ decreases) the theorem follows.

THEOREM 4.4. *Let $\mu \in M(R)$, $\{r_t\}_{t\geqslant 0} \subset (0, 1)$ with $\lim_{t\to\infty} r_t = 0$, then the following properties are equivalent*:

(a) $\rho_t(\theta, \mu) = O(r_t)$, $t \to \infty$.

(b) *There exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$, there exists*

$$d\mu = |h_t(x)|\, e^{u_t(x)}\, dx,$$

where $h_t \in H^1(R)$ and $v_t = -\arg(e_{-t}\bar{\theta}h_t)$ satisfy

$$\|v_t\|_\infty = O(r_t), \qquad \left\|\frac{\cosh|u_t|}{\cos v_t} - 1\right\|_\infty = O(r_t^2), \qquad t\to\infty.$$

Proof. (a) $\rightarrow$ (b) There exist $c, \delta > 0$ such that $\pi/2 - \arccos x < cx$ if $0 < x < \delta$.

If $0 < x^2 < 1/2$ then $1/\sqrt{1-x^2} \leqslant 1 + x^2$.

There exist s and t_1 such that $\rho_t(\theta, \mu) \leqslant sr_t < 1$ if $t \geqslant t_1$.

From Theorem 4.2, μ has the required representation,

$$|u_t(x)| \leqslant \operatorname{arccosh}\left(\frac{\cos v_t(x)}{\sqrt{1-s^2r_t^2}}\right), \qquad |v_t(x)| \leqslant \frac{\pi}{2} - \arccos sr_t < \frac{\pi}{2}.$$

But there is t_2 such that if $t \geqslant t_2$, $sr_t < \delta$ and $sr_t < 1/\sqrt{2}$. So if we take $t_0 = \max\{t_1, t_2\}$ then, for every $t \geqslant t_0$

$$\frac{\cosh |u_t|}{\cos v_t} \leqslant \frac{1}{\sqrt{1-s^2r_t^2}} \leqslant 1 + s^2r_t^2, \qquad |v_t(x)| \leqslant \int_0^{sr_t} \frac{1}{\sqrt{1-z^2}}\, dz = O(r_t).$$

(b) $\rightarrow$ (a) There exist $\lambda > 0$, $\delta \in (0, 1)$ such that

$$\lambda x < \pi/2 - \arccos x \qquad \text{if} \quad 0 < x < \delta.$$

Assume there are c and t_0 such that for $t \geqslant t_0$, μ has the representation with

$$\|v_t\|_\infty \leqslant cr_t \qquad \text{and} \qquad \frac{\cosh |u_t|}{\cos v_t} \leqslant 1 + c^2r_t^2.$$

Taking into account that for all $x \in R$, with $x^2 < \frac{1}{2}$, $1 + x^2 \leqslant 1/\sqrt{1-2x^2}$ we obtain for sufficiently large t

$$\frac{\cosh |u_t|}{\cos v_t} \leqslant \frac{1}{\sqrt{1-2c^2r_t^2}}.$$

We can take t such that $cr_t/\lambda < 1$ and $2c^2r_t^2 < 1$ (because $r_t \rightarrow 0$.)

There exist t_1 such that if $t \geqslant t_1$, $cr_t/\lambda < \delta$.

Let $t' = \max\{t_0, t_1\}$; thus for every $t \geqslant t'$

$$|v_t(x)| \leqslant cr_t \leqslant \pi/2 - \arccos(cr_t/\lambda).$$

We also have

$$\cosh |u_t(x)| \leqslant (1 + c^2r_t^2) \cos v_t(x) \leqslant \frac{\cos v_t(x)}{\sqrt{1-2c^2r_t^2}}.$$

Let $d = \max\{c/\lambda; \sqrt{2c^2}\}$ then $dr_t < 1$. Hence,

$$|v_t(x)| \leqslant \arcsin dr_t \qquad \text{and} \qquad |u_t(x)| \leqslant \operatorname{arccosh}\left(\frac{\cos v_t(x)}{\sqrt{1-d^2r_t^2}}\right).$$

From Theorem 4.2, $\rho_t(\theta, \mu) \leqslant dr_t$ for every $t \geqslant t'$.

The last result gives a necessary and sufficient condition for the rate of convergence of $\rho_t(\theta, \mu)$. The next theorem gives a necessary and another sufficient condition which are simpler than the one given in Theorem 4.4.

THEOREM 4.5. *Let* $\mu \in M(R)$, $\{r_t\}_{t \geqslant 0} \subset (0, 1)$ *with* $\lim_{t \to \infty} r_t = 0$. *We have the following properties*:

(1) *Suppose there exists* $t_0 \geqslant 0$ *such that for all* $t \geqslant t_0$, *there exists*

$$h_t \in H^1(R) \qquad \textit{with} \quad d\mu = |h_t(x)|\, e^{u_t(x)}\, dx$$

and $v_t = -\arg(e_{-t}\bar{\theta}h_t)$ *satisfies*

$$\|v_t\|_\infty = O(r_t^2) \qquad \textit{and} \qquad \|u_t\|_\infty = O(r_t), \quad t \to \infty.$$

Then $\rho_t(\theta, \mu) = O(r_t)$, $t \to \infty$.

(2) *If* $\rho_t(\theta, \mu) = O(r_t)$, $t \to \infty$, *then there exists* $t_0 \geqslant 0$ *such that for all* $t \geqslant t_0$, *there exists*

$$h_t \in H^1(R) \qquad \textit{with} \quad d\mu = |h_t(x)|\, e^{u_t(x)}\, dx$$

and $v_t = -\arg(e_{-t}\bar{\theta}h_t)$ *satisfying*

$$\|v_t\|_\infty = O(r_t^2) \qquad \textit{and} \qquad \|u_t\|_\infty = O(r_t), \quad t \to \infty.$$

Proof. There exist $c > 0$, $d > 0$, and $\delta \in (0, 1)$ such that if $0 < x < \delta$:

(i) $cx < \pi/2 - \arccos x < dx$, (ii) $cx < \operatorname{arccosh} \sqrt{1 + x^2} < dx$,

(iii) $\dfrac{1}{\sqrt{1 - x^2}} < d$, (iv) $\dfrac{x}{\sqrt{1 - x^2}} < \delta$.

(1) By hypothesis there exist t_0 and k (we can take $k > c$) such that if $t \geqslant t_0$,

$$\|u_t\|_\infty \leqslant kr_t, \qquad \|v_t\|_\infty \leqslant kr_t^2, \qquad \text{and} \qquad kr_t/c < \delta.$$

So by (i)

$$\|v_t\|_\infty \leqslant kr_t^2 \leqslant c\left(\frac{kr_t}{c}\right)^2 < \frac{\pi}{2} - \arccos\left(\frac{kr_t}{c}\right)^2 < \frac{\pi}{2} - \arccos\left(\frac{kr_t}{c}\right).$$

This also implies that

$$\cos v_t \geqslant \cos\left(\frac{\pi}{2} - \arccos\left(\frac{kr_t}{c}\right)^2\right) = \sin \arccos\left(\frac{kr_t}{c}\right)^2$$
$$= \sqrt{1 - \left(\frac{kr_t}{c}\right)^4}.$$

Using (ii) we obtain

$$\begin{aligned}
&\operatorname{arccosh}\left(\frac{\cos v_t}{\sqrt{1-(kr_t/c)^2}}\right)\\
&\qquad\geqslant \operatorname{arccosh}\left(\frac{\sqrt{1-(kr_t/c)^4}}{\sqrt{1-(kr_t/c)^2}}\right)\\
&\qquad= \operatorname{arccosh}\sqrt{1+(kr_t/c)^2} \geqslant c(kr_t/c) = kr_t \geqslant \|u_t\|_\infty.
\end{aligned}$$

By Theorem 4.2, $\rho_t(\theta, \mu) \leqslant kr_t/c$ if $t \geqslant t_0$.

(2) If there are t_0 and k such that $\rho_t(\theta, \mu) \leqslant kr_t < \delta$ for $t \geqslant t_0$, applying Theorem 4.2, and from (i), we have

$$\|v_t\|_\infty \leqslant \pi/2 - \arccos kr_t \leqslant dk\, r_t.$$

From (iv), (ii), and (iii)

$$\begin{aligned}
\|u_t\|_\infty &\leqslant \operatorname{arccosh}\left(\frac{\cos v_t}{\sqrt{1-k^2r_t^2}}\right) \leqslant \operatorname{arccosh}\left(\frac{1}{\sqrt{1-k^2r_t^2}}\right)\\
&= \operatorname{arccosh}\sqrt{1+\left(\frac{kr_t}{\sqrt{1-k^2r_t^2}}\right)^2} \leqslant d\left(\frac{kr_t}{\sqrt{1-k^2r_t^2}}\right) \leqslant d^2k\, r_t.
\end{aligned}$$

5. Characterization of the Generalized Maximal Correlation Coefficient for Tempered Measures

Definition. For $b \geqslant 0$. A measure μ is tempered of order $\leqslant b$ if $\mu/(x^2+1)^{b/2}$ is a finite measure.

$$M^b(R) = \{\text{positive Borel tempered measures of order } \leqslant b \text{ in } R\}$$

$$E_{1,b} = \{f: f(x) = (x+i)^{-b/2}\, \phi(x),\ \phi \in E_1\},$$

$$E_{2,b} = \{f: f(x) = (x-i)^{-b/2}\, \phi(x),\ \phi \in E_2\}.$$

We have that $E_{1,b} \subset L^2(R, \mu)$ and $E_{2,b} \subset L^2(R, \mu)$ if $\mu \in M^b(R)$.
For $\mu \in M^b(R)$, we define

$$\rho_{t;b}(\mu) = \sup_{\phi_1 \in E_{1;b};\ \phi_2 \in E_{2;b};\ \int_{-\infty}^{\infty} |\phi_1|^2\, d\mu = \int_{-\infty}^{\infty} |\phi_2|^2\, d\mu = 1} \left| \int_{-\infty}^{\infty} e_t \phi_1 \overline{\phi_2}\, d\mu \right|.$$

We shall study the following problems:

(1) Characterization of the $\mu \in M^b(R)$ such that $\rho_{t,b}(\mu) \leqslant r$ for $r \in (0, 1)$.

(2) Characterization of the $\mu \in M^b(R)$ such that $\lim_{t\to\infty} \rho_{t,b}(\mu) = 0$.

(3) Characterization of the $\mu \in M^b(R)$ such that $\rho_{t,b}(\mu) = O(r_t)$ with given $r_t \to 0$ $(t \to \infty)$.

For the proof of the results of this section consider the function $\theta(x) = ((x-i)/(x+i))^{b/2}$, and use the results of Section 4 taking into account that

(a) $\mu \in M^b(R)$ if and only if $\mu/(x^2+1)^{b/2} \in M(R)$;

(b) $\rho_{t,b}(\mu) = \rho_t(\theta, \mu/(x^2+1)^{b/2})$;

(c) $\arg((x+i)^b) = \arg(\bar{\theta}(x))$.

PROPOSITION 5.1. *Let $\mu \in M^b(R)$ and $0 < r < 1$.*

$$\begin{bmatrix} \dfrac{r\mu}{(x^2+1)^{b/2}} & \dfrac{e_t\mu}{(x+i)^b} \\ \dfrac{e_{-t}\mu}{(x-i)^b} & \dfrac{r\mu}{(x^2+1)^{b/2}} \end{bmatrix}$$

is weakly positive if and only if $\rho_{t,b}(\mu) \leqslant r$.

In this case, μ is an absolutely continuous measure.

The answer to Problems (1) and (2) are given by the next results, which are generalizations to the continuous case and to tempered measures of order $\leqslant b$ of the theorems of Helson and Sarason [12] and Sarason [16]. In the case $t = 0$, Corollary 5.2 is a generalization of the theorem of Helson and Szegö [13]. These theorems study the positivity of the angle between past and future, in the discrete case.

COROLLARY 5.2. *Let b, $t \geqslant 0$, $\mu \in M^b(R)$, and $0 < r < 1$; then the following properties are equivalent:*

(a) $\rho_{t,b}(\mu) \leqslant r$.

(b) *There exists $h \in H^1(R)$ such that $d\mu = (x^2+1)^{b/2}\, |h(x)|\, e^{u(x)}\, dx$ and $v(x) = -\arg((x+i)^b\, e_{-t}(x)\, h(x))$ satisfy*

$$|u(x)| \leqslant \operatorname{arccosh}\left(\frac{\cos v(x)}{\sqrt{1-r^2}}\right), \qquad |v(x)| \leqslant \frac{\pi}{2} - \arccos r < \frac{\pi}{2}.$$

COROLLARY 5.3. *Let $\mu \in M^b(R)$, then the following properties are equivalent:*

(a) $\lim_{t\to\infty} \rho_{t,b}(\mu) = 0$.

(b) *For every $\varepsilon > 0$, there exists $t > 0$ and $h_\varepsilon \in H^1(R)$ such that $d\mu = (x^2+1)^{b/2}\, |h_\varepsilon(x)|\, e^{u_\varepsilon(x)}\, dx$, $v_\varepsilon(x) = -\arg((x+i)^b\, e_{-t}(x)\, h_\varepsilon(x))$, and $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$.*

The following results characterize the rate of convergence of $\rho_{t,b}(\mu)$.

COROLLARY 5.4. *Let* $\mu \in M^b(R)$, $\{r_t\}_{t \geq 0} \subset (0, 1)$ *with* $\lim_{t \to \infty} r_t = 0$, *then the following properties are equivalent*:

(a) $\rho_{t,b}(\mu) = O(r_t)$, $t \to \infty$.

(b) *There exists* $t_0 \geq 0$ *such that for all* $t \geq t_0$,

$$d\mu = (x^2+1)^{b/2} \, |h_t(x)| \, e^{u_t(x)} \, dx,$$

where $h_t \in H^1(R)$ *and* $v_t(x) = -\arg((x+i)^b \, e_{-t}(x) \, h_t(x))$ *satisfy*

$$\|v_t\|_\infty = O(r_t), \qquad \frac{\cosh |u_t|}{\cos v_t} - 1 = O(r_t^2), \quad t \to \infty.$$

COROLLARY 5.5. *Let* $\mu \in M^b(R)$, $\{r_t\}_{t \geq 0} \subset (0, 1)$ *with* $\lim_{t \to \infty} r_t = 0$. *We have the following properties*:

(1) *If there exists* $t_0 \geq 0$ *such that for all* $t \geq t_0$, *there exists*

$$h_t \in H^1(R) \qquad d\mu = (x^2+1)^{b/2} \, |h_t(x)| \, e^{u_t(x)} \, dx$$

and $v_t(x) = -\arg((x+i)^b \, e_{-t}(x) \, h_t(x))$ *satisfies*

$$\|v_t\|_\infty = O(r_t^2) \qquad \textit{and} \qquad \|u_t\|_\infty = O(r_t), \quad t \to \infty,$$

then $\rho_{t,b}(\mu) = O(r_t)$, $t \to \infty$.

(2) *If* $\rho_{t,b}(\mu) = O(r_t)$, $t \to \infty$, *then there exists* $t_0 \geq 0$ *such that for all* $t \geq t_0$, *there exists*

$$h_t \in H^1(R) \qquad d\mu = (x^2+1)^{b/2} \, |h_t(x)| \, e^{u_t(x)} \, dx$$

and $v_t(x) = -\arg((x+i)^b \, e_{-t}(x) \, h_t(x))$ *satisfy*

$$\|v_t\|_\infty = O(r_t^2) \qquad \textit{and} \qquad \|u_t\|_\infty = O(r_t), \quad t \to \infty.$$

6. CHARACTERIZATION OF THE COMPLETELY REGULAR PROCESSES

We improve our results for the case $b = 0$ showing that the weights can be expressed in terms of an entire function of exponential type. An argument of Hayashi [11] forms the basis for the proof of these results which were given in [5]; here we will obtain them from the results of Section 5.

Let G be the family of entire functions [3] Γ of exponential type which are bounded on the real axis and zero free in the upper halfplane.

Let $\tilde{v}$ denote the harmonic conjugate of v.

LEMMA 6.1. $w(x) = |h(x)| \, e^{u(x)}$, *where* $h \in H^1(R)$, u *and* v *are real* *bounded functions on R and $v(x) = -\arg(e_{-t}(x)\,h(x))$ for a $t>0$ and $|v(x)| < \pi/2$ if and only if*

$$w(x) = |g(x)|^2 = (x^2+1)\,|\Gamma(x)|^2 \exp(u(x) + \tilde{v}(x))$$

and $|v(x)| < \pi/2$, where $g \in H^2(R)$ is outer, u and v are real bounded functions on R, and $\Gamma \in G$.

For the proof of this lemma, see [5, 8].

Let $\{X_t\}_{t\in R}$ be a weakly stationary process with spectral measure $\mu \in M(R)$ and $E(X_t) = 0$. The completely regular coefficient or maximal correlation coefficient $\rho_t(\mu)$ of $\{X_t\}_{t\in R}$ is the cosine of the angle between the past and the future, that is, $\rho_t(\mu) = \rho_{t,0}(\mu) = \rho_t(1, \mu)$. For Gaussian processes this is the mixing coefficient.

DEFINITION. A weakly stationary process is said to be completely linearly regular if (see [15])

$$\lim_{t\to\infty} \rho_t(\mu) = 0.$$

For the Gaussian case these are the strongly mixing processes.

COROLLARY 6.2. *Let $\mu \in M(R)$, $r \in (0, 1)$, $t \geq 0$ then the following properties are equivalent:*

(a) $\rho_t(\mu) \leq r$.

(b) $d\mu = |g(x)|^2\,dx = (x^2+1)\,|\Gamma(x)|^2 \exp(u(x) + \tilde{v}(x))\,dx$, *where $g \in H^2(R)$ is outer, u and v are real bounded functions on R, $\Gamma \in G$,*

$$|u(x)| \leq \operatorname{arccosh}\left(\frac{\cos v(x)}{\sqrt{1-r^2}}\right), \qquad |v(x)| \leq \frac{\pi}{2} - \arccos r < \frac{\pi}{2}.$$

The following results characterizes the completely linearly regular processes. For the Gaussian case this is a characterization of the strongly mixing processes.

COROLLARY 6.3. *Let $\mu \in M(R)$, then the following properties are equivalent:*

(a) $\lim_{t\to\infty} \rho_t(\mu) = 0$.

(b) *For every $\varepsilon > 0$, there exist $g \in H^2(R)$ outer, u_ε and v_ε real bounded functions on R, and $\Gamma_\varepsilon \in G$, such that*

$$d\mu = |g(x)|^2\,dx = (x^2+1)\,|\Gamma_\varepsilon(x)|^2 \exp(u_\varepsilon(x) + \tilde{v}_\varepsilon(x))\,dx$$

and $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$.

Now we will give results for the rate of convergence of the maximal correlation coefficient.

COROLLARY 6.4. *Let $\mu \in M(R)$, $\{r_t\}_{t \geqslant 0} \subset (0, 1)$ with $\lim_{t \to \infty} r_t = 0$, then the following properties are equivalent*:

(a) $\rho_t(\mu) = O(r_t)$, $t \to \infty$.

(b) *There exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$, there exists $g \in H^2(R)$ outer, u_t and v_t real bounded functions on R, and $\Gamma_t \in G$, such that*

$$d\mu = |g(x)|^2\, dx = (x^2+1)\, |\Gamma_t(x)|^2 \exp(u_t(x) + \tilde{v}_t(x))\, dx$$

$$\|v_t\|_\infty = O(r_t), \qquad \frac{\cosh |u_t|}{\cos v_t} - 1 = O(r_t^2), \quad t \to \infty.$$

COROLLARY 6.5. *Let $\mu \in M(R)$ and $\{r_t\}_{t \geqslant 0} \subset (0, 1)$ with $\lim_{t \to \infty} r_t = 0$. We have the following properties*:

(1) *If there exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$, there exists $g \in H^2(R)$ outer, u_t and v_t real bounded functions on R, and $\Gamma_t \in G$, such that*

$$d\mu = |g(x)|^2\, dx = (x^2+1)\, |\Gamma_t(x)|^2 \exp(u_t(x) + \tilde{v}_t(x))\, dx$$

$$\|v_t\|_\infty = O(r_t^2) \qquad \textit{and} \qquad \|u_t\|_\infty = O(r_t), \quad t \to \infty,$$

then $\rho_t(\mu) = O(r_t)$, $t \to \infty$.

(2) *If $\rho_t(\mu) = O(r_t)$, $t \to \infty$, then there exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$, there exists $g \in H^2(R)$ outer, u_t and v_t real bounded functions on R, and $\Gamma_t \in G$, such that*

$$d\mu = |g(x)|^2\, dx = (x^2+1)\, |\Gamma_t(x)|^2 \exp(u_t(x) + \tilde{v}_t(x))\, dx$$

$$\|v_t\|_\infty = O(r_t^2) \qquad \textit{and} \qquad \|u_t\|_\infty = O(r_t), \quad t \to \infty.$$

7. INTERPOLATION

The coefficient $\rho_t(\theta, \mu)$ can also be used for solving the interpolation problem. The problem for interpolation for width $2t > 0$ is to compute the distribution of $X_{s'}$, for fixed $|s'| < t$ conditional upon X_s, $|s| \geqslant t$. This problem received its first full solution in the book of Dym and McKean [10].

Let $\mu = \mu_s + w(x)\, dx$ be the spectral measure of the process. In trigonometric language we have to project the exponential $e_{s'}$ onto

$$E^t = \text{the span in } L^2(\mu) \text{ of } e_s : |s| \geqslant t.$$

For fixed $t > 0$ the interpolation is perfect if $E^t = L^2(\mu)$. The interpolation is imperfect if $E^t \neq L^2(\mu)$.

It is natural to take $w = |g|^2$ for $g \in H^2$, where g is an outer function because otherwise the problem is trivial, in fact, if

$$\int_{-\infty}^{\infty} \frac{w(x)}{x^2+1}\, dx = -\infty,$$

$E^t = L^2(\mu)$. As well we can take $\mu_s = 0$, because the singular space is contained in E^t and may be dispensed with, you may as well take $d\mu = w(x)\, dx$. For the refined versions of Szegö's alternative due to Krein and Wiener, see [10].

THEOREM 7.1 [9]. *Let θ be the function $\theta(x) = (x+i)/(x-i)$. Let $t > 0$ and μ the spectral measure of a process with spectral density $w = |g|^2$ where g is an outer function of H^2. Then the following conditions are equivalent*:

(a) *The interpolation is imperfect for t.*

(b) *There exist $r \in (0, 1)$ such that $\rho_{2t}(\theta, \mu) \leqslant r$.*

(c) *There exist $r \in (0, 1)$, $h \in H^1$, u and v bounded real functions such that*

$$w = |h|\, e^u \qquad v = -\arg(e_{-2t}\bar{\theta} h)$$

$$\|v\|_\infty \leqslant \frac{\pi}{2} - \arccos r < \frac{\pi}{2} \qquad |u| \leqslant \operatorname{arccosh}\left(\frac{\cos v}{\sqrt{1-r^2}}\right).$$

(d) *There exist $r \in (0, 1)$, $g \in H^2$ outer, u and v real bounded functions on R, such that*

$$w(x) = |g(x)|^2 = (x^2+1)\, |\Gamma(x)|^2 \exp(u(x) + \tilde{v}(x)),$$

where $\Gamma \in G$ and

$$\|v\|_\infty \leqslant \frac{\pi}{2} - \arccos r < \frac{\pi}{2} \qquad |u| \leqslant \operatorname{arccosh}\left(\frac{\cos v}{\sqrt{1-r^2}}\right).$$

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